

ON COMMON ZEROS OF EIGENFUNCTIONS OF THE LAPLACE OPERATOR

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ABSTRACT. We consider the eigenfunctions of the Laplace operator Δ on a compact Riemannian manifold M of dimension n . For M homogeneous with irreducible isotropy representation and for a fixed eigenvalue λ of Δ we find the average number of common zeros of n eigenfunctions. It turns out that, up to a constant depending on n , this number equals $\lambda^{n/2} \text{vol } M$, the expression known from the celebrated Weyl's law. To prove this we compute the volume of the image of M under an equivariant immersion into a sphere.

1. INTRODUCTION

Let M be a compact Riemannian manifold without boundary, $n = \dim M$. Fix an eigenvalue λ of the Laplace operator Δ on M , such that the corresponding eigenspace

$$W_\lambda = \{u \in C^\infty(M, \mathbb{R}) \mid \Delta u + \lambda u = 0\}$$

has sufficiently large dimension. Moreover, assume that for some eigenfunctions $u_1, \dots, u_n \in W_\lambda$ the set of common zeros

$$Z(u_1, \dots, u_n) = \{x \in M \mid u_1(x) = \dots = u_n(x) = 0\}$$

is finite. As we will see later, this happens quite often for homogeneous spaces of compact Lie groups. Our goal is to evaluate the number of points in $Z(u_1, \dots, u_n)$.

The zero set of an eigenfunction is called a nodal set. The connected components of its complement are called nodal domains. According to the classical Courant's theorem [3], the number of nodal domains determined by k -th eigenfunction is at most k , see also [2], [8]. For the standard sphere S^2 the eigenfunctions are spherical harmonics. A spherical harmonic of degree m has eigenvalue $\lambda_m = m(m+1)$ of multiplicity $2m+1$, so the number of nodal domains is smaller than or

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equal to $k = m^2 + 1$. Therefore the nodal set has at most m^2 connected components. On the other hand, it follows from Bézout's theorem that the number of points in $Z(u_1, u_2)$ for two generic spherical harmonics of degree m does not exceed $2m^2$, see Sect. 4. Now, if u_1 is fixed, one can define u_2 as a small perturbation of u_1 by a rotation. Let p be the number of connected components of the nodal set $u_1 = 0$. Take one of them and observe that it intersects some connected component of the nodal set $u_2 = 0$. Moreover, by Jordan theorem applied to a loop of the former connected component it follows that the number of intersection points is at least 2. Altogether, we get $2p$ points in the zero set $Z(u_1, u_2)$, proving the Courant's estimate $p \leq m^2$.

Motivated by this example, one is led to the problem of estimating $\#Z(u_1, \dots, u_n)$ from above. Such an estimate is a simple result for the sphere, see Theorem 4.1. The proof is based on the relation between spherical harmonics and homogeneous harmonic polynomials on the ambient vector space, so there is no generalization to arbitrary M . Alternatively, for M homogeneous and isotropy irreducible we find the average number of common zeros of eigenfunctions. In [1], V. Arnold suggested to study the topology of the set of common zeros for $m \leq n$ eigenfunctions, see Problem 2003-10, p.174. Under our assumptions, we consider this problem for $m = n$.

We now state our main result. Let K be a connected compact Lie group, $L \subset K$ a closed subgroup, and $M = K/L$. Fix a K -invariant Riemannian metric g on M . In what follows, we consider real valued functions on M , for which the scalar product is given by

$$(f_1, f_2) = \int_M f_1(x) f_2(x) dx,$$

where dx is the Riemannian measure. Since the Laplace operator Δ on M is K -invariant, we have an orthogonal representation of K in the eigenspace W_λ . From now on we tacitly assume that $\lambda > 0$. Let $\{0\} \neq H \subset W_\lambda$ be a K -invariant subspace of dimension N .

For $x \in M$ we have the linear functional $\alpha_x \in H^*$, defined by $\alpha_x(u) = u(x)$, where $u \in H$. Take an orthonormal basis $\{f_i\}_{i=1}^N$ of H and identify H^* with \mathbb{R}^N via

$$(\mu_1, \dots, \mu_N) \mapsto \mu_1 f_1^* + \dots + \mu_N f_N^*,$$

where $f_i^*(f_j) = \delta_{ij}$. Then

$$\alpha_x = \sum f_i(x) f_i^*$$

and so the map $M \rightarrow H^*$, $x \mapsto \alpha_x$, is written as

$$f = (f_1, \dots, f_N) : M \rightarrow \mathbb{R}^N.$$

Clearly, f is equivariant with respect to the given action of K on M and the linear representation of K in H^* identified with \mathbb{R}^N .

We now restrict our attention to the class of isotropy irreducible homogeneous spaces. This means that the subgroup $L \subset K$ acts in the tangent space to M via an irreducible representation. Isotropy irreducible homogeneous spaces are listed in [7] and [10]. We note that symmetric spaces of simple groups belong to this class.

In what follows, we denote by σ_n the n -dimensional volume of the unit sphere $S^n \subset \mathbb{R}^{n+1}$. We write $Z(U)$ for the set of common zeros of all functions from an n -dimensional subspace $U \subset H$. If u_1, \dots, u_n is a basis of U then $Z(U) = Z(u_1, \dots, u_n)$. The average $\mathfrak{M}\{\#Z(U)\}$ is taken over all n -dimensional subspaces U of H , i.e., over the Grassmanian $\text{Gr}_n(H)$, with respect to the action of $\text{SO}(N, \mathbb{R})$, endowed with the Haar measure of volume 1, on the set of these subspaces.

Theorem 1.1. *Let $M = K/L$ be isotropy irreducible and let $H \subset W_\lambda$ be a K -invariant subspace. Then*

$$\mathfrak{M}\{\#Z(U)\} = \frac{2}{\sigma_n} \left(\frac{\lambda}{n}\right)^{n/2} \text{vol } M = c_n \lambda^{n/2} \text{vol } M,$$

where c_n depends only on the dimension of M . In particular, for $M = S^n$ and for the eigenvalue $\lambda = m(m+n-1)$ one has

$$\mathfrak{M}\{\#Z(U)\} = 2 \left(\frac{m(m+n-1)}{n}\right)^{n/2}.$$

We remark that in the latter case the representation in W_λ is irreducible, i.e., $H = W_\lambda$.

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2. EQUIVARIANT MAPPINGS AND INDUCED METRICS

We use the notations introduced in Sect. 1. The scalar product $g(\xi, \eta)$ on a tangent space $T_x(M)$ defines a scalar product on the dual space $T_x^*(M)$, denoted again by g . Furthermore, for tangent vectors and covectors we write (ξ, η) instead of $g(\xi, \eta)$ and $\|\xi\|^2$ instead of $g(\xi, \xi)$.

Theorem 2.1. *One has*

$$\sum_{i=1}^N \|df_i\|^2 = \frac{\lambda N}{\text{vol } M}$$

everywhere on M .

Proof. Fix $k \in K$ and put $f_i^*(x) = f_i(k^{-1}x)$. Then

$$f_i^*(x) = \sum_j a_{ij} f_j(x), \quad df_i^*(x) = \sum_j a_{ij} df_j(x),$$

where (a_{ij}) is an orthogonal matrix. Therefore

$$||df_i^*(x)||^2 = \sum_{p,q} a_{ip} a_{iq} \cdot (df_p(x), df_q(x)),$$

hence

$$\sum_i ||df_i(x)||^2 = \sum_i ||df_i^*(x)||^2.$$

Now, for $\xi \in T_x M$ and $\eta = (dk)\xi \in T_{kx} M$ one has

$$df_i^*(kx)(\eta) = df_i(x)(\xi),$$

where $\xi \in T_x M$ and $\eta = (dk)\xi \in T_{kx} M$. Taking the maximum over the unit sphere in $T_x M$, we get

$$||df_i^*(kx)||^2 = ||df_i(x)||^2.$$

Since k is arbitrary, it follows that $\sum_i ||df_i(x)||^2 = D$, where D is a constant. To find D write

$$\operatorname{div}(f_i \operatorname{grad} f_i) = f_i \Delta f_i + ||\operatorname{grad} f_i||^2 = -\lambda f_i^2 + ||df_i||^2.$$

Summing up and integrating over M , we obtain

$$\lambda N = \lambda \sum_i \int_M f_i^2 dx = D \cdot \operatorname{vol} M,$$

hence

$$D = \frac{\lambda N}{\operatorname{vol} M}.$$

□

Lemma 2.2. *Let l_k , $k = 1, \dots, N$ be linear forms in n variables t_j , such that $\sum l_k^2 = C \cdot \sum t_j^2$. Then $\sum ||l_k||^2 = Cn$, where $||l|| := \sqrt{\sum b_j^2}$ for $l = \sum b_j t_j$.*

Proof. Write $l_k = \sum b_{kj} t_j$. Then $\sum_k b_{ki} b_{kj} = C \cdot \delta_{ij}$ by assumption. In particular,

$$\sum_{k=1}^N ||l_k||^2 = \sum_k \sum_j b_{kj}^2 = \sum_j \sum_k b_{kj}^2 = Cn.$$

□

For the convenience of the reader we give the proofs of some known results. In quantum mechanics, the next theorem for spherical harmonics is called Unsöld's theorem (1927). The general case is also found in the literature, see [5], Exercise 5.25 c), i), p. 261 and p. 303.

Theorem 2.3. *One has*

$$\sum_{i=1}^N f_i^2 = R^2,$$

where $R = \sqrt{\frac{N}{\text{vol } M}}$, i.e., $f(M)$ is contained in the sphere $S^{N-1}(R) \subset \mathbb{R}^N$ of radius R .

Proof. Define $f_i^*(x)$ as in the proof of Theorem 2.1. The same argument using the orthogonality of (a_{ij}) shows that

$$\sum_i f_i(x)^2 = \sum_i f_i^*(x)^2.$$

By transitivity of K on M it follows that

$$\sum_{i=1}^N f_i^2 = R^2,$$

where R^2 is a constant. Integrating over M yields $R^2 \cdot \text{vol } M = N$. \square

Let \mathfrak{k} and \mathfrak{l} be the Lie algebras of K and, respectively, L . If $M = K/L$ is isotropy irreducible, then the L -module $\mathfrak{k}/\mathfrak{l}$, defined by the adjoint action of L , is irreducible. In particular, an L -invariant intermediate Lie subalgebra between \mathfrak{l} and \mathfrak{k} coincides with one of these two algebras. Therefore, for a subgroup $L_1 \subset K$ containing L , we have one of the two possibilities: either $L \subset L_1$ is a finite extension, or $L_1 = K$.

The first assertion in the following theorem is known, see [5], Exercise 5.25 c), ii), p. 261 and p. 303.

Theorem 2.4. *Assume that the isotropy representation of L in the tangent space to M is irreducible. Then:*

- 1) *the mapping $f : M \rightarrow f(M)$ is a covering of some degree d and a local isometry up to a dilation;*
- 2) *the inverse image of Euclidean metric on \mathbb{R}^N is given by*

$$f^*\left(\sum dt_i^2\right) = \frac{\lambda N}{n \text{vol } M} \cdot g$$

and the volume of $f(M) \subset S^{N-1}(R)$ is equal to

$$\text{vol } f(M) = \frac{1}{d} \cdot \left(\frac{\lambda N}{n \text{vol } M}\right)^{n/2} \cdot \text{vol } M.$$

Proof. 1) Recall that $f : M \rightarrow \mathbb{R}^N$ is an equivariant map with respect to the action of K on the homogeneous space M and a linear representation of K in \mathbb{R}^N . Therefore $f(M)$ is one orbit of K in \mathbb{R}^N . In particular, $f(M)$ is a manifold and a homogeneous space.

By our assumption, an equivariant mapping from M to any homogeneous space of K is either a covering map or the map to one point. But $\lambda > 0$, the functions f_i are non-constant, and so f is a covering map. By the irreducibility of the isotropy representation it follows that there is only one up to a scalar factor K -invariant metric on M . Hence

$$f^*(\sum dt_i^2) = C \cdot g.$$

2) By Lemma 2.2 and Theorem 2.1 we have

$$Cn = \sum \|df_i\|^2 = \frac{\lambda N}{\text{vol } M}.$$

The measure on M induced by the metric $C \cdot g$ is $C^{n/2} \cdot dx$, and so we obtain the expression for $\text{vol } f(M)$.

□

3. THE KINEMATIC FORMULA

The kinematic formula is proven by L.A.Santaló in his book [9], see Sect.15.2 and 18.6 therein. Another proof is given by R.Howard, see [6], Sect. 3.12. The formula is valid for any space of constant curvature, but we will use it only for the sphere. Also, we will need only the special case when two submanifolds of the sphere have complementary dimensions. In geometric probability, the results of this type are called Crofton formulae. Let M and L be two submanifolds of the sphere $S^{N-1} \subset \mathbb{R}^N$, $n = \dim M$, $l = \dim L$, and $n + l = N - 1$. For the applications it is convenient to consider the sphere of radius R . The kinematic density is given by the Haar measure dg on $\text{SO}(N, \mathbb{R})$. The kinematic formula is the equality

$$\int_{M \cap g \cdot L \neq \emptyset} \#(M \cap g \cdot L) \cdot dg = C (\text{vol } M) (\text{vol } L),$$

where C is a constant independent of M and L . The constant is easy to find. Namely, draw two planes of dimensions $n + 1$ and $l + 1$ through the origin and take for M and L the plane sections. They are isometric to spheres of dimensions n and l , and the number of intersection points is 2 almost everywhere. Therefore

$$C = \frac{2}{\sigma_n \sigma_l R^{N-1}}.$$

We will apply the kinematic formula for arbitrary M and for a plane section L of complementary dimension through the origin. The kinematic formula means that the average number of intersection points of M by such a plane is equal to

$$\int_{M \cap g \cdot L \neq \emptyset} \#(M \cap g \cdot L) \cdot dg = \frac{2 \operatorname{vol} M}{\sigma_n R^n}.$$

Proof of Theorem 1.1. We have the covering $f : M \rightarrow f(M) \subset S^{N-1}$ of degree d . The set of common zeros for an n -dimensional subspace $U \subset H \subset W_\lambda$ can be written as

$$Z(U) = f^{-1}(f(M) \cap L)$$

for some L . If this set is finite then

$$\#Z(U) = d \cdot \#(f(M) \cap L).$$

For the average number of zeros the kinematic formula gives

$$\begin{aligned} \mathfrak{M}\{\#Z(U)\} &= d \cdot \int_{f(M) \cap g \cdot L \neq \emptyset} \#(f(M) \cap g \cdot L) dg = \\ &= \frac{2d \cdot \operatorname{vol} f(M)}{\sigma_n R^n} = \frac{2}{\sigma_n} \left(\frac{\lambda}{n}\right)^{n/2} \cdot \operatorname{vol} M, \end{aligned}$$

where the last equality follows from Theorems 2.3 and 2.4, 2).

□

4. UPPER BOUND IN THE CASE $M = S^n$

Let $M = S^n$ be the unit sphere in \mathbb{R}^{n+1} . An eigenfunction of the Laplace operator on M is the restriction of a homogeneous harmonic polynomial $p(t_1, \dots, t_{n+1})$. The polynomial is uniquely defined by the maximum principle for harmonic functions. If $m = \deg p$ then $u = p|_{S^n}$ satisfies $\Delta u + \lambda u = 0$, where $\lambda = m(m+n-1)$. Let $u_i = p_i|_{S^n}$, $\deg p_i = m_i$, $i = 1, 2, \dots, n$. We call u_1, \dots, u_n generic if p_1, \dots, p_n have finitely many common zeros on the complex affine hypersurface $\sum z_i^2 = 1$ in \mathbb{C}^{n+1} .

Theorem 4.1. *Let u_1, \dots, u_n be a generic system of spherical harmonics. Then*

$$\#Z(u_1, \dots, u_n) \leq 2 \cdot m_1 \cdot \dots \cdot m_n.$$

Proof. Consider the imbedding $\mathbb{C}^{n+1} \subset \mathbb{P}^{n+1}(\mathbb{C})$ and denote by X the intersection of hypersurfaces

$$p_1(t_1, \dots, t_{n+1}) = 0, \dots, p_n(t_1, \dots, t_{n+1}) = 0, \quad t_1^2 + \dots + t_{n+1}^2 = s^2$$

in $\mathbb{P}^{n+1}(\mathbb{C})$ with homogeneous coordinates $(t_1 : \dots : t_{n+1} : s)$. Then we have the inclusion of finite sets

$$X(\mathbb{R}) = Z(u_1, \dots, u_n) \subset X \cap \mathbb{C}^{n+1}.$$

It follows that the number of irreducible components of X is greater than or equal to $\#Z(u_1, \dots, u_n)$. On the other hand, by Bézout's theorem this number does not exceed the product of degrees $2 \cdot \prod m_i$, see [4], Example 8.4.6. \square

5. CONCLUDING REMARKS

1. Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be n not necessarily distinct positive eigenvalues of Δ . For each λ_i choose a subspace $H_i \subset W_{\lambda_i}$ as in Sect. 1, put $N_i = \dim H_i$, and consider the equivariant map $f^{(i)} : M \rightarrow S^{N_i-1}$. Then the map

$$f : M \rightarrow S^{N_1-1} \times \dots \times S^{N_n-1}, \quad x \mapsto (f^{(1)}(x), \dots, f^{(n)}(x))$$

is equivariant with respect to the direct sum of representations of K . For $u_1 \in H_1, \dots, u_n \in H_n$ one can mimick our definition in Sect. 1 to get the average number of zeros $\mathfrak{M}\{\#Z(u_1, \dots, u_n)\}$. This amounts to averaging the number of intersection points of $f(M)$ with the products $L_1 \times \dots \times L_n$, where L_i is a hyperplane section of S^{N_i-1} through the origin. It is then plausible to conjecture that $\mathfrak{M}\{\#Z(u_1, \dots, u_n)\}$ equals

$$\frac{2\sqrt{\lambda_1 \dots \lambda_n}}{\sigma_n n^{n/2}} \cdot \text{vol } M,$$

generalizing Theorem 1.1. Unfortunately, since there is no kinematic formula for the product of spheres, the proof in Sect. 3 does not work.

2. It would be interesting to have a lower bound for $\#Z(u_1, \dots, u_n)$ at least for $M = S^n$. For $M = S^2$ one can easily construct two spherical harmonics of degree m , such that $\#Z(u_1, u_2) = 2m$. Namely, let v_m be the so called zonal spherical harmonic, i.e., the spherical harmonic invariant under the group of rotations around a given axis. The zeros of v_m are located on m circles orthogonal to the rotation axis. Take another axis at angular distance α from the given one and denote by v_m^α the associate zonal spherical harmonic of degree m . The zeros of v_m^α are located on m circles, orthogonal to the new rotation axis. If α is small enough then each circle in $v_m^\alpha = 0$ intersects exactly one circle in $v_m = 0$ in two points. Thus $\#Z(v_m, v_m^\alpha) = 2m$.

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